Mathematics 1 AESB1110: Exam 3

November 3, 2014

Question 1: Compute the integrals total 2 p.

$$\int \cos^3(x)\,dx$$
$$\int \sin^2\left(\frac{\theta}{2}\right)\,d\theta$$

Answer:

$$\begin{aligned}\int \cos^3(x)\,dx &= \int \cos^2(x)\cos(x)\,dx = \int \left(1-\sin^2(x)\right)\cos(x)\,dx\\ &= \begin{bmatrix} u=\sin(x) \\ du=\cos(x)\,dx\end{bmatrix} = \int \left(1-u^2\right)\,du\\ &= u-\frac{1}{3}u^3+C = \sin(x)-\frac{1}{3}\sin^3(x)+C\end{aligned}$$

1 p.

$$\begin{aligned}\int \sin^2\left(\frac{\theta}{2}\right)\,d\theta &= \int \frac{1}{2}\left(1-\cos\left(2\frac{\theta}{2}\right)\right)\,d\theta = \frac{1}{2}\int \left(1-\cos(\theta)\right)\,d\theta\\ &= \frac{1}{2}\theta-\frac{1}{2}\sin(\theta)+C\end{aligned}$$

1 p.

Question 2: Compute the integrals total 2 p.

$$\int \frac{x^2}{x^2-4}\,dx$$
$$\int \frac{1}{x^3+4x}\,dx$$

Answer:

$$\int \frac{x^2}{x^2-4}\,dx = \begin{bmatrix} \frac{x^2}{x^2-4} = 1 + \frac{4}{x^2-4} \\ = 1 + \frac{4}{(x-2)(x+2)} \\ = 1 + \frac{A}{x-2} + \frac{B}{x+2} \\ A = 1, \quad B = -1 \end{bmatrix} = \int \left(1 + \frac{1}{x-2} - \frac{1}{x+2}\right) dx$$

$$= x + \ln|x-2| - \ln|x+2| + C = x + \ln\left|\frac{x-2}{x+2}\right| + C$$

1 p.

$$\int \frac{1}{x^3+4x}\,dx = \begin{bmatrix} \frac{1}{x^3+4x} = \frac{1}{x(x^2+4)} \\ D = 0^2 - 4(1)(4) = -16 < 0 \\ \frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \\ A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = 0 \end{bmatrix}$$

$$= \int \left(\frac{1/4}{x} - \frac{x/4}{x^2+4}\right) dx = \frac{1}{4}\ln|x| - \frac{1}{4}\int \frac{x\,dx}{x^2+4}$$

$$= \begin{bmatrix} u = x^2+4 \\ du = 2x\,dx \end{bmatrix} = \frac{1}{4}\ln|x| - \frac{1}{8}\int \frac{du}{u}$$

$$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln|u| + C = \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) + C$$

1 p.

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total 2 p.

Question 3 (a): At what points does the curve $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$ intersect the surface $x^2 + y^2 + z^2 = 5$

Answer: From $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$ we deduce that

$$x(t) = \sin(t), \quad y(t) = \cos(t), \quad z(t) = t$$

Substitution in the equation of the surface and solving for t:

$$\begin{aligned} \sin^2(t) + \cos^2(t) + t^2 &= 5 \\ 1 + t^2 &= 5 \\ t^2 &= 4 \\ t_1 = 2, \quad t_2 &= -2. \end{aligned}$$

Computing the (x, y, z)-coordinates of the two points: 1 p.

$$P_1(x(t_1), y(t_1), z(t_1)) = P_1(\sin(2), \cos(2), 2)$$
$$P_2(x(t_2), y(t_2), z(t_2)) = P_2(\sin(-2), \cos(-2), -2)$$

Question 3 (b): Find a vector function that represents the curve of intersection of the surfaces $z = 4x^2 + y^2$ and $y = x^2$

Answer: Choosing parametrization as $x = t$. Then,

$$x = t, \quad y = t^2, \quad z = 4t^2 + t^4,$$
$$\mathbf{r}(t) = \langle t, t^2, 4t^2 + t^4 \rangle$$

Alternatively, choosing $y = t$:

$$x = \sqrt{t}, \quad y = t, \quad z = 4t + t^2,$$
$$\mathbf{r}(t) = \langle \sqrt{t}, t, 4t + t^2 \rangle, \quad t \geq 0.$$

1 p.

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total 2 p.

Question 4 (a): Suppose that the vector function $\mathbf{r}(t) = \langle \sin(2\pi t), \cos(2\pi t), t \rangle$, $t \geq 0$ represents the trajectory of an object as a function of time. Compute the velocity $\mathbf{v}(10)$ of this object at $t = 10$.

Answer: The velocity $\mathbf{v}(t)$ is the time-derivative of the position. Hence,

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2\pi\cos(2\pi t), -2\pi\sin(2\pi t), 1 \rangle$$
$$\mathbf{v}(10) = \langle 2\pi\cos(20\pi), -2\pi\sin(20\pi), 1 \rangle = \langle 2\pi, 0, 1 \rangle$$

1 p.

Question 4 (b): Suppose that the vector function $\mathbf{v}(t) = \langle \sin(2\pi t), \cos(2\pi t), t \rangle$, $t \geq 0$ represents the velocity of an object as a function of time. Compute the position $\mathbf{r}(10)$ of this object at $t = 10$, if the initial position at $t = 0$ is $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$.

Answer: To find position from velocity we compute: 1 p.

$$\begin{aligned}
\mathbf{r}(10) &= \mathbf{r}(0) + \int_0^{10} \mathbf{v}(t)\, dt \\
&= \langle 0, 0, 0 \rangle + \left\langle -\frac{1}{2\pi}\cos(2\pi t), \frac{1}{2\pi}\sin(2\pi t), \frac{1}{2}t^2 \right\rangle \Big|_0^{10} \\
&= \left\langle -\frac{1}{2\pi}\cos(20\pi), \frac{1}{2\pi}\sin(20\pi), \frac{1}{2}10^2 \right\rangle - \left\langle -\frac{1}{2\pi}\cos(0), \frac{1}{2\pi}\sin(0), \frac{1}{2}0^2 \right\rangle \\
&= \left\langle -\frac{1}{2\pi}, 0, 50 \right\rangle - \left\langle -\frac{1}{2\pi}, 0, 0 \right\rangle = \langle 0, 0, 50 \rangle
\end{aligned}$$

total 2 p.

Question 5 (a): Consider the object with the trajectory $\mathbf{r}(t) = \langle \sin(2\pi t), \cos(2\pi t), t \rangle$, $t \geq 0$. What arc length s has the object traveled between $t = 0$ and $t = 10$?

Answer:

$$s = \int_0^{10} |\mathbf{r}'(t)| \, dt = \int_0^{10} |\langle 2\pi \cos(2\pi t), -2\pi \sin(2\pi t), 1 \rangle| \, dt$$
$$= \int_0^{10} \sqrt{(2\pi \cos(2\pi t))^2 + (-2\pi \sin(2\pi t))^2 + 1^2} \, dt = \int_0^{10} \sqrt{4\pi^2 + 1} \, dt$$
$$= t\sqrt{4\pi^2 + 1}\Big|_0^{10} = 10\sqrt{4\pi^2 + 1}$$

1 p.

Question 5 (b): Reparametrize the above trajectory in terms of the traveled arc length s instead of t.

Answer:

$$s(t) = \int_0^{t} |\mathbf{r}'(u)| \, du = u\sqrt{4\pi^2 + 1}\Big|_0^{t} = t\sqrt{4\pi^2 + 1}$$
$$t = \frac{s}{\sqrt{4\pi^2 + 1}}$$
$$\mathbf{r}(s) = \left\langle \sin\left(\frac{2\pi s}{\sqrt{4\pi^2 + 1}}\right), \cos\left(\frac{2\pi s}{\sqrt{4\pi^2 + 1}}\right), \frac{s}{\sqrt{4\pi^2 + 1}} \right\rangle$$

1 p.